

# F-SINGULARITIES UNDER GENERIC LINKAGE

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*Dedicated to Prof. Craig Huneke on the occasion of his 65th birthday*

ABSTRACT. Let  $R = k[x_1, \dots, x_n]$  be a polynomial ring over a perfect field of positive characteristic. Let  $I$  be an unmixed ideal in  $R$  and let  $J$  be a generic link of  $I$  in  $S = R[u_{ij}]_{c \times r}$ . We describe the parameter test submodule of  $S/J$  in terms of the test ideal of the pair  $(R, I)$  when a reduction of  $I$  is a complete intersection or almost complete intersection. As an application, we deduce a criterion for when  $S/J$  has  $F$ -rational singularities in these cases. We also compare the  $F$ -pure threshold of  $(R, I)$  and  $(S, J)$ .

## 1. INTRODUCTION

Let  $R = k[x_1, \dots, x_n]$  be a polynomial ring over a field of positive characteristic. Let  $I = (f_1, \dots, f_r)$  be an unmixed ideal in  $R$  of height  $c$ , where unmixed means that all associated primes of  $I$  have the same height [Mat86]. We can define a regular sequence  $g_1, \dots, g_c$  in  $S = R[u_{ij}]_{c \times r}$  via  $g_i := u_{i1}f_1 + \dots + u_{ir}f_r$ , where the  $u_{ij}$  are variables over  $S$ . Then  $J = (g_1, \dots, g_c) : I$  is called a generic link of  $I$  in  $S = R[u_{ij}]$ . The study of generic linkage has attracted considerable attention and has been developed widely from both algebraic and geometric points of view [HU87], [HU88], [CU02], [EHU04], [Niu14].

In contrast to the quick and deep development of singularity theories in the past decades, much less has been known about the behaviors of singularities under generic linkage. A special case is a result of Chardin and Ulrich [CU02] which says that if  $R/I$  is a complete intersection and has rational (resp.  $F$ -rational) singularities, then a generic link  $S/J$  also has rational (resp.  $F$ -rational) singularities. This result in characteristic zero has been vastly extended in recent work of Niu [Niu14], which is our main motivation for this research.

**Theorem 1.1** (Theorem 1.1 in [Niu14]). *Let  $J$  be a generic link of a reduced and equidimensional ideal  $I$  in  $S = R[u_{ij}]$  and assume that the characteristic of  $k$  is 0. We have*

- (1)  $\omega_{S/J}^{GR} \cong \mathcal{J}(R, I^c) \cdot (S/J)$ , where  $\omega_{S/J}^{GR}$  denotes the Grauert-Riemenschneider canonical sheaf of  $S/J$  and  $\mathcal{J}(R, I^c)$  denotes the multiplier ideal of the pair  $(R, I^c)$ ,
- (2)  $\text{lct}(S, J) \geq \text{lct}(R, I)$ . In particular, if the pair  $(R, I^c)$  is log canonical, then the pair  $(S, J^c)$  is also log canonical.

This result gives a nice criterion for a generic link to have rational singularities in characteristic 0. It also has applications to bounding the Castelnuovo-Mumford regularity of projective varieties [Niu14, Corollary 1.2]. Since test ideals and  $F$ -pure thresholds are characteristic  $p$  analogues of multiplier ideals and log canonical thresholds (c.f. [BST15] and [HY03]), it is natural to ask whether analogues of Theorem 1.1 hold for test ideals and  $F$ -pure thresholds. Our main result is the following, which partially extends Theorem 1.1 to characteristic  $p$  and generalizes [CU02, Theorem 3.13] in characteristic  $p$ .

**Theorem 1.2** (Theorem 3.3, Corollary 4.4). *Let  $J$  be a generic link of an unmixed ideal  $I$  in  $S = R[u_{ij}]$  and assume the characteristic of  $k$  is  $p > 0$ .*

- (1) *Suppose  $I$  is reduced and that a reduction of  $I$  is a complete intersection or an almost complete intersection. Then  $\tau(\omega_{S/J}) \cong \tau(R, I^c) \cdot (S/J)$ , where  $\tau(\omega_{S/J})$  denotes the parameter test submodule and  $\tau(R, I^c)$  denotes the test ideal of the pair  $(R, I^c)$ .*
- (2) *Suppose that a reduction of  $I$  is a complete intersection. Then  $\text{fpt}_S(J) \geq \text{fpt}_R(I)$ . In particular, if the pair  $(R, I^c)$  is  $F$ -pure, then the pair  $(S, J^c)$  is also  $F$ -pure.*

This paper is organized as follows: in Section 2 we recall and prove some basic result for  $F$ -singularities and test ideals; in Section 3 we give a description of the parameter test submodule of  $S/J$  in terms of the test

ideal of the pair  $(R, I)$ , when a reduction of  $I$  is a complete intersection or an almost complete intersection. This generalizes earlier results in [CU02]. In Section 4 we compare the  $F$ -pure threshold of the pairs  $(S, J)$  and  $(R, I)$  when a reduction of  $I$  is a complete intersection.

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## 2. $F$ -SINGULARITIES AND TEST IDEALS

In this section we collect some basic definitions of  $F$ -singularities and test ideals and prove a characteristic  $p > 0$  analogue of Ein's Lemma in [Niu14], which will be used in later sections.

Let  $R$  be a noetherian commutative ring of characteristic  $p > 0$ . We will use  $F_*^e R$  to denote the target of the  $e$ -th Frobenius endomorphism  $F^e : R \xrightarrow{r \mapsto r^{p^e}} R$ , i.e.  $F_*^e R$  is an  $R$ -bimodule, which is the same as  $R$  as an abelian group and as a right  $R$ -module, that acquires its left  $R$ -module structure via the  $e$ -th Frobenius endomorphism  $F^e : R \xrightarrow{r \mapsto r^{p^e}} R$ . When  $R$  is reduced, we will use  $R^{1/p^e}$  to denote the ring whose elements are  $p^e$ -th roots of elements of  $R$ . Note that these notations (when  $R$  is reduced)  $F_*^e R$  and  $R^{1/p^e}$  are used interchangeably in the literature; we will do so in this paper as well assuming no confusion will arise.

*Remark 2.1.* If  $R$  is a commutative ring essentially of finite type over a perfect field of characteristic  $p > 0$ , then  $R$  admits a canonical module denoted by  $\omega_R$ . Applying  $\text{Hom}_R(-, \omega_R)$  to the  $e$ -th Frobenius  $R \rightarrow F_*^e R$  produces an  $R$ -linear map

$$\text{Hom}_R(F_*^e R, \omega_R) \rightarrow \text{Hom}_R(R, \omega_R) = \omega_R.$$

Moreover, we have  $F_*^e \omega_R \cong \text{Hom}_R(F_*^e R, \omega_R)$  (see [BST15, Example 2.4] for more details). Hence we have a natural  $R$ -linear map:

$$\Phi_R^e : F_*^e \omega_R \cong \text{Hom}_R(F_*^e R, \omega_R) \rightarrow \text{Hom}_R(R, \omega_R) = \omega_R$$

called the trace map of the  $e$ -th Frobenius.

**Example 2.2.** When  $R = k[x_1, \dots, x_n]$  is a polynomial ring over a perfect field  $k$  of characteristic  $p > 0$ , we can identify  $\omega_R$  with  $R$ , and  $\Phi_R^e$  can be identified with the usual trace  $\text{Tr}_R^e$ , that is:

$$\text{Tr}_R^e(F_*^e(x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n})) = \begin{cases} x_1^{\frac{i_1-(p^e-1)}{p^e}} x_2^{\frac{i_2-(p^e-1)}{p^e}} \cdots x_n^{\frac{i_n-(p^e-1)}{p^e}}, & \text{if } \frac{i_t-(p^e-1)}{p^e} \in \mathbb{Z} \text{ for each } t \\ 0, & \text{otherwise} \end{cases}$$

In this case  $\text{Hom}_R(F_*^e R, R)$  is a cyclic  $F_*^e R$ -module generated by  $\text{Tr}_R^e$ . Furthermore, if  $f_1, \dots, f_c$  is a regular sequence in  $R$  and  $T = R/(f_1, \dots, f_c)$ , then we have ([Fed83, Corollary on page 465])

$$\Phi_T^e(F_*^e(-)) = \text{Tr}_R^e(F_*^e(f_1^{p^e-1} \cdots f_c^{p^e-1} \cdot -)).$$

**Lemma 2.3.** *Let  $S \rightarrow R$  be a surjection of noetherian commutative rings of characteristic  $p$ . Assume that both  $S$  and  $R$  admit canonical module  $\omega_S$  and  $\omega_R$  respectively and  $\dim S = \dim R$ . Then*

$$\Phi_R^e = \Phi_S^e|_{\omega_R}.$$

*Proof.* Under our assumptions, we have  $\omega_R = \text{Hom}_S(R, \omega_S)$  and the surjection  $S \rightarrow R$  induces an inclusion  $\omega_R = \text{Hom}_S(R, \omega_S) \hookrightarrow \omega_S$ . Consider the following diagram

$$\begin{array}{ccccc} \text{Hom}_R(F_*^e R, \text{Hom}_S(R, \omega_S)) & \longrightarrow & \text{Hom}_R(R, \text{Hom}_S(R, \omega_S)) & \xrightarrow{\sim} & \text{Hom}_S(R, \omega_S) \\ \downarrow & & \downarrow & & \downarrow \\ \text{Hom}_S(F_*^e S, \omega_S) & \longrightarrow & \text{Hom}_S(S, \omega_S) & \xrightarrow{\sim} & \omega_S \end{array}$$

Note that the top row (resp. the bottom row) induces the trace map  $\Phi_R^e$  (resp.  $\Phi_S^e$ ). To prove our lemma, it suffices to prove

- (a) the vertical map on the left is an inclusion, and
- (b) the diagram commutes

To prove (a), note that the vertical map on the left can be refined further as

$$\begin{aligned}
\mathrm{Hom}_R(F_*^e R, \mathrm{Hom}_S(R, \omega_S)) &= \mathrm{Hom}_S(F_*^e R, \mathrm{Hom}_S(R, \omega_S)) \\
&\hookrightarrow \mathrm{Hom}_S(F_*^e S, \mathrm{Hom}_S(R, \omega_S)) \text{ since } F_*^e S \twoheadrightarrow F_*^e R \\
&\hookrightarrow \mathrm{Hom}_S(F_*^e S, \omega_S) \text{ since } \mathrm{Hom}_S(R, \omega_S) \hookrightarrow \omega_S
\end{aligned}$$

To prove (b), note that the commutativity follows directly from the commutativity of

$$\begin{array}{ccc}
S & \longrightarrow & F_*^e S \\
\downarrow & & \downarrow \\
R & \longrightarrow & F_*^e R
\end{array}$$

□

**Definition 2.4** (cf. Definition 2.33 in [BST15]). Let  $R$  be an  $F$ -finite noetherian integral domain of characteristic  $p$ . The *parameter test submodule*  $\tau(\omega_R)$  is the unique smallest nonzero submodule  $M$  of  $\omega_R$  such that  $\Phi_R(F_* M) \subseteq M$ .  $R$  is called *F-rational* if  $R$  is Cohen-Macaulay and  $\tau(\omega_R) = \omega_R$ . Note that this is not the original definition of  $F$ -rationality, but is known to be equivalent [Smi97].

**Definition 2.5** (cf. Definition 3.16 and Theorem 3.18 in [Sch11]). Let  $R$  be an  $F$ -finite noetherian integral domain of characteristic  $p$ . Let  $I \subseteq R$  be a nonzero ideal and  $t \in \mathbb{Q}_{\geq 0}$ . We define the *test ideal*  $\tau(R, I^t)$ , abbreviated  $\tau(I^t)$ , to be the unique smallest nonzero ideal  $J \subseteq R$  such that  $\phi(F_*^e(I^{\lceil t(p^e-1) \rceil} J)) \subseteq J$  for all  $e > 0$  and all  $\phi \in \mathrm{Hom}_R(F_*^e R, R)$ .

**Definition 2.6** (cf. Definitions 1.3 and 2.1 and Proposition 2.2 in [TW04]). Let  $R$  be an  $F$ -finite noetherian integral domain of characteristic  $p$ . Let  $I \subset R$  be an ideal and  $t \geq 0$  be a real number.

- (1) The pair  $(R, I^t)$  is *F-pure* if for all large  $e \gg 0$ , there exists an element  $d \in I^{\lceil t(p^e-1) \rceil}$  such that  $F_*^e dR \hookrightarrow F_*^e R$  splits as an  $R$ -module homomorphism.
- (2) The pair  $(R, I^t)$  is *strongly F-regular* if for every  $c \neq 0$  there exists  $e \geq 0$  and  $d \in I^{\lceil tp^e \rceil}$  such that  $F_*^e(cd)R \hookrightarrow F_*^e R$  splits as an  $R$ -module homomorphism.
- (3) The *F-pure threshold*  $\mathrm{fpt}_R(I)$  of  $(R, I)$  is  $\sup\{s \in \mathbb{R}_{\geq 0} \mid \text{the pair } (R, I^s) \text{ is } F\text{-pure}\}$ , and we have also  $\mathrm{fpt}_R(I) = \sup\{s \in \mathbb{R}_{\geq 0} \mid \text{the pair } (R, I^s) \text{ is strongly } F\text{-regular}\}$ .

*Remark 2.7.* Note that  $(R, I^t)$  is strongly  $F$ -regular if and only if  $\tau(I^t) = R$ . Indeed, suppose  $(R, I^t)$  is strongly  $F$ -regular. Pick a nonzero element  $c \in J$  and take  $e \gg 0$  and  $d \in I^{\lceil tp^e \rceil}$  satisfying the conditions of strong  $F$ -regularity for  $c$ , and let  $\phi : F_*^e R \rightarrow R$  be a map such that  $\phi(F_*^e(cd)) = 1$ . Then

$$\phi(F_*^e(I^{\lceil t(p^e-1) \rceil} J)) \supseteq \phi(F_*^e(I^{\lceil tp^e \rceil} J)) = R,$$

and so  $\tau(I^t) = R$ .

On the other hand, if  $\tau(I^t) = R$ ,  $0 \neq c \in R$ , and  $a \in I^{\lceil t \rceil}$ , then there exists  $e \geq 0$  and  $\phi : F_*^e R \rightarrow R$  such that  $\phi(F_*^e(I^{\lceil t(p^e-1) \rceil} acR)) = R$ . Let  $b \in I^{\lceil t(p^e-1) \rceil}$  and  $f \in R$  such that  $\phi(F_*^e(cabf)) = 1$ . Then we are done once we note that  $abf \in I^{\lceil t \rceil} I^{\lceil t(p^e-1) \rceil} \subseteq I^{\lceil tp^e \rceil}$ .

We will need the following important description of test ideals:

**Theorem 2.8** (cf. Proof of Theorem 3.18 in [Sch11]). *With the notations as in Definition 2.5, for any nonzero  $a \in \tau(I^t)$ , we have:*

$$\tau(I^t) = \sum_{e \geq 0} \sum_{\phi} \phi(F_*^e(aI^{\lceil t(p^e-1) \rceil}))$$

where the inner sum runs over all  $\phi \in \mathrm{Hom}_R(F_*^e R, R)$ .

*Remark 2.9.* With the notations as in Definition 2.5, the following holds ([BSTZ10, 3.3])

$$(2.9.1) \quad \tau(I^t) = \sum_{e \geq 0} \sum_{\phi \in \mathrm{Hom}_R(F_*^e R, R)} \phi(F_*^e(dI^{\lceil tp^e \rceil}))$$

where  $d$  is a big test element.

If  $R = k[x_1, \dots, x_n]$  is a polynomial ring over a perfect field  $k$  of characteristic  $p > 0$ , then one can set  $d = 1$  in (2.9.1) and  $\text{Hom}_R(F_*^e R, R)$  is a cyclic  $F_*^e R$ -module generated by  $\text{Tr}_R^e$  as discussed earlier. Hence by (2.9.1),

$$\begin{aligned} \tau(I^t) &= \sum_{e \geq 0} \sum_{\phi \in \text{Hom}_R(F_*^e R, R)} \phi(F_*^e(aI^{\lceil t(p^e-1) \rceil})) = \sum_{e \geq 0} \text{Tr}_R^e(F_*^e(aI^{\lceil t(p^e-1) \rceil})), \text{ for any } a \in \tau(I^t) \\ &= \sum_{e \geq 0} \sum_{\phi \in \text{Hom}_R(F_*^e R, R)} \phi(F_*^e(I^{\lceil tp^e \rceil})) = \sum_{e \geq 0} \text{Tr}_R^e(F_*^e(I^{\lceil tp^e \rceil})) \end{aligned}$$

*Remark 2.10.* With the notations as in Definition 2.4, one can show that if  $R_{a'}$  is regular, then for every sufficiently large power  $a$  of  $a'$ ,  $\tau(\omega_R) = \sum_e \Phi_R^e(F_*^e(a \cdot \omega_R))$ . This can be proved by a similar argument as [ST12, Lemma 3.6, Lemma 3.8] so we omit the details.

The following result from [ST12] will also be used.

**Lemma 2.11** (cf. Theorem 6.9 in [ST12]). *Let  $R$  be an integral domain essentially of finite type over a perfect field of characteristic  $p > 0$  and let  $I, J \subseteq R$  be nonzero ideals and  $t \in \mathbb{R}_{\geq 0}$ .*

- (1) *If  $J$  is a reduction of  $I$ , then  $\tau(I^t) = \tau(J^t)$ .*
- (2) *If  $J$  is generated by  $r$  elements, then  $\tau(J^r) = J\tau(J^{r-1})$ .*

We are ready to prove the characteristic  $p > 0$  analogue of Ein's Lemma in [Niu14]:

**Lemma 2.12** (Ein's Lemma in characteristic  $p > 0$ ). *Let  $R$  be an integral domain essentially of finite type over a perfect field of characteristic  $p > 0$  and let  $I \subseteq R$  be an unmixed ideal of codimension  $c$ . If  $\tau(I^{c-1}) = R$ , then  $\tau(I^c) = I$ . In particular, if  $R$  is strongly  $F$ -regular and  $(R, I^c)$  is  $F$ -pure, then  $\tau(I^c) = I$ .*

*Proof.* The lemma will follow from the following two inclusions:

$$(2.12.1) \quad \tau(I^c) \subseteq I.$$

$$(2.12.2) \quad I\tau(I^{t-1}) \subseteq \tau(I^t) \text{ for all } t \geq 1.$$

Indeed, if  $\tau(I^{c-1}) = R$ , then  $I = I\tau(I^{c-1}) \subseteq \tau(I^c) \subseteq I$ , and so we have equality throughout.

*Proof of (2.12.1).* Since inclusion is a local condition, we may assume that  $R$  is local with maximal ideal  $\mathfrak{m}$ . By replacing  $R$  by  $R[x]_{\mathfrak{m}R[x]}$ , we may assume that  $R$  has infinite residue field: it is straightforward to check that  $\tau(I^c)R[x]_{\mathfrak{m}R[x]} = \tau((IR[x]_{\mathfrak{m}R[x]})^c)$ . Now let  $\mathfrak{p}$  be a minimal prime of  $I$ . Since  $I$  is unmixed,  $\dim R_{\mathfrak{p}} = c$ . Hence  $IR_{\mathfrak{p}}$  has a reduction  $J \subseteq IR_{\mathfrak{p}}$  generated by  $c$  elements. Therefore, since test ideals localize,

$$\tau(I^c)R_{\mathfrak{p}} = \tau((IR_{\mathfrak{p}})^c) = \tau(J^c) = J\tau(J^{c-1}) \subseteq J \subseteq IR_{\mathfrak{p}}.$$

Since every associated prime of  $I$  is minimal, this inclusion holds for all associated primes of  $I$ , hence it holds globally, i.e.  $\tau(I^c) \subseteq I$ .  $\square$

*Proof of (2.12.2).* Let  $t \in \mathbb{R}_{\geq 1}$ , and pick  $0 \neq a \in \tau(I^t)$ . Then

$$\begin{aligned} I\tau(I^{t-1}) &= I \sum_{e \geq 0} \sum_{\phi} \phi \left( F_*^e \left( aI^{\lceil (t-1)(p^e-1) \rceil} \right) \right) \\ &= \sum_{e \geq 0} \sum_{\phi} \phi \left( F_*^e \left( aI^{\lceil p^e \rceil} I^{\lceil (t-1)(p^e-1) \rceil} \right) \right) \\ &\subseteq \sum_{e \geq 0} \sum_{\phi} \phi \left( F_*^e \left( aI^{p^e} I^{\lceil (t-1)(p^e-1) \rceil} \right) \right) \\ &= \sum_{e \geq 0} \sum_{\phi} \phi \left( F_*^e \left( aI^{p^e + \lceil (t-1)(p^e-1) \rceil} \right) \right) \\ &\subseteq \sum_{e \geq 0} \sum_{\phi} \phi \left( F_*^e \left( aI^{\lceil tp^e \rceil} \right) \right) \\ &= \tau(I^t), \end{aligned}$$

where the inner sum runs over all  $\phi \in \text{Hom}_R(F_*^e R, R)$  and the last inclusion following from the fact that

$$p^e + \lceil (t-1)(p^e - 1) \rceil = \lceil p^e + (t-1)(p^e - 1) \rceil = \lceil t(p^e - 1) + 1 \rceil > \lceil t(p^e - 1) \rceil. \quad \square$$

For the last statement, if  $(R, I^c)$  is  $F$ -pure, then the  $F$ -pure threshold of  $I$  is at least  $c$ . Since the  $F$ -pure threshold is the supremum of those values  $t$  for which  $(R, I^t)$  is strongly  $F$ -regular when  $R$  is strongly  $F$ -regular [TW04, Proposition 2.2], we have that  $(R, I^{c-1})$  is strongly  $F$ -regular. This means that  $\tau(I^{c-1}) = R$  by Remark 2.7, and hence the first statement of the lemma tells us  $\tau(I^c) = I$ .  $\square$

### 3. $F$ -RATIONALITY UNDER GENERIC LINKAGE

In this section, we investigate how  $F$ -singularities (e.g.  $F$ -purity,  $F$ -rationality, etc) behave under a generic linkage. To this end, we will also consider the behaviors of test ideals under a generic linkage. We begin with recalling the definition of a generic link.

**Definition 3.1.** Let  $R = k[x_1, \dots, x_n]$  be a polynomial ring over a perfect field of positive characteristic. Let  $I$  be an unmixed ideal of  $R$  of height  $c$ . Fix a generating set  $\{f_1, \dots, f_r\}$  of  $I$ . Let  $(u_{ij})$ ,  $1 \leq i \leq c$ ,  $1 \leq j \leq r$ , be a  $c \times r$  matrix of variables. Consider  $c$  elements  $g_1, \dots, g_c$  in  $S = R[u_{ij}]$  defined by

$$g_i := u_{i1}f_1 + u_{i2}f_2 + \dots + u_{ir}f_r$$

for  $1 \leq i \leq c$ . Then  $J = (g_1, \dots, g_c) : (IS)$  is called the *first generic link* of  $I$  with respect to  $\{f_1, \dots, f_r\}$  (we also call  $S/J$  the generic link of  $R/I$  with respect to  $\{f_1, \dots, f_r\}$ ).

*Remark 3.2.* It is well known that under the above assumptions, if  $I$  is reduced, then  $IS$  and  $J$  are *geometrically linked*, i.e.,  $IS = (g_1, \dots, g_c) : J$  and  $IS \cap J = (g_1, \dots, g_c)$ . Moreover,  $J$  is actually a prime ideal of height  $c$  [HU85, 2.6].

The following theorem is our main technical result in this section.

**Theorem 3.3.** *With the notation as in Definition 3.1, assuming  $I$  is reduced, we have*

- (1)  $\tau(\omega_{S/J}) \subseteq \tau(I^c) \cdot (S/J)$ ;
- (2) *If  $I$  has a minimal reduction generated by at most  $c+1$  elements, then  $\tau(\omega_{S/J}) \supseteq \tau(I^c) \cdot (S/J)$ ; hence  $\tau(\omega_{S/J}) = \tau(I^c) \cdot (S/J)$  in this case.*

Our proof of Theorem 3.3(2) requires considering different sets of generators of  $I$ . A priori, a generic link  $(S, J)$  depends on the choice of generators. The following lemma guarantees that the statement in Theorem 3.3(2) is independent of the choice of generators of  $I$ . Its proof follows the same line as the proof of [HU87, Proposition 2.11].

**Lemma 3.4.** *Let  $\Lambda_1$  and  $\Lambda_2$  be two sets of generators of  $I$  and let  $(S_1, J_1)$  and  $(S_2, J_2)$  be generic links of  $I$  with respect to  $\Lambda_1$  and  $\Lambda_2$  respectively. Then  $\tau(\omega_{S_1/J_1}) \supseteq \tau(I^c) \cdot (S_1/J_1)$  iff  $\tau(\omega_{S_2/J_2}) \supseteq \tau(I^c) \cdot (S_2/J_2)$ .*

*Proof.* By considering  $\Lambda_1 \cup \Lambda_2$ , we can assume that  $\Lambda_1 \subseteq \Lambda_2$ . By induction on the difference between the cardinality of  $\Lambda_1$  and  $\Lambda_2$ , we may assume that  $\Lambda_2$  has one more element than  $\Lambda_1$ , i.e. we may assume that  $\Lambda_1 = \{f_1, \dots, f_r\}$  and  $\Lambda_2 = \Lambda_1 \cup \{f_{r+1}\}$ .

Denote the height of  $I$  by  $c$ . Let  $\{u_{ij} \mid 1 \leq i \leq c, 1 \leq j \leq r+1\}$  be indeterminates over  $R$ . Set  $S_1 = R[u_{i,j}]_{1 \leq i \leq c, 1 \leq j \leq r}$  and  $S_2 = R[u_{i,j}]_{1 \leq i \leq c, 1 \leq j \leq r+1}$ . For  $i = 1, \dots, c$ , set

$$g_i := u_{i,1}f_1 + \dots + u_{i,r}f_r$$

and

$$h_i := u_{i,1}f_1 + \dots + u_{i,r+1}f_{r+1}.$$

Then  $J_1 = ((g_1, \dots, g_c) :_S IS)$  is the first generic link of  $I$  with respect to  $\Lambda_1$  and  $J_2 = ((h_1, \dots, h_c) :_{S_2} IS_2)$  is the first generic link of  $I$  with respect to  $\Lambda_2$ .

It is clear that  $S_2 = S_1[u_{1,r+1}, \dots, u_{c,r+1}]$ . Since  $f_{r+1} \in I$ , we must have that  $f_{r+1} = \sum_{j=1}^r a_j f_j$  for some  $a_j \in R$ . Let  $\varphi : S_2 \rightarrow S_2$  be the automorphism given by the linear change of variables

$$u_{i,j} \mapsto u_{i,j} + u_{i,r+1}a_j$$

for  $1 \leq i \leq c$  and  $1 \leq j \leq r$  and

$$u_{i,r+1} \mapsto u_{i,r+1}$$

for  $1 \leq i \leq c$ .

We claim that  $\varphi(J_1 S_2) = J_2$  and we reason as follows. For  $i = 1, \dots, c$ , we have that

$$\varphi(g_i) = \sum_{j=1}^r (u_{i,j} + u_{i,r+1} a_j) f_j = \sum_{j=1}^r u_{i,j} f_j + u_{i,r+1} \sum_{j=1}^r a_j f_j = \sum_{j=1}^r u_{i,j} f_j + u_{i,r+1} f_{r+1} = h_i.$$

Now since  $S_1 \hookrightarrow S_2$  is a faithfully flat extension, we have that

$$J_1 S_2 = ((g_1, \dots, g_c) :_{S_1} I S_1) S_2 = ((g_1, \dots, g_c) S_2 :_{S_2} I S_2),$$

and hence

$$\varphi(J_1 S_2) = \varphi((g_1, \dots, g_c) S_2 :_{S_2} I S_2) = (\varphi(g_1, \dots, g_c) S_2 :_{S_2} \varphi(I S_2)) = ((h_1, \dots, h_c) :_{S_2} I S_2) = J_2.$$

Let  $S_2^\varphi$  denote the  $S_1$ -algebra that is the same as  $S_2$  as a ring and whose  $S_1$ -module structure is induced by  $S_1 \hookrightarrow S_2 \xrightarrow{\varphi} S_2$ . Then we have shown that  $J_1 \otimes_{S_1} S_2^\varphi = J_2$  and hence  $S_1/J_1 \otimes_{S_1} S_2^\varphi = S_2/J_2$ . It is straightforward to check that

$$\tau(\omega_{S_1/J_1}) \otimes_{S_1} S_2^\varphi = \tau(\omega_{S_1/J_1} \otimes_{S_1} S_2^\varphi) = \tau(\omega_{S_2/J_2}).$$

Our lemma follows immediately since  $S_2^\varphi$  is faithfully flat over  $S_1$ .  $\square$

The following lemma is also needed in the proof of Theorem 3.3.

**Lemma 3.5.** *Let  $c, r$  be positive integers such that  $c = r$  or  $c = r - 1$ . Let  $\beta = (\beta_1, \dots, \beta_r)$  be an element of  $\mathbb{N}^r$ , where  $\mathbb{N}$  is the set of non-negative integers. Assume  $\sum_i \beta_i = c(p^e - 1)$ . Then there exist  $c$  elements  $\alpha_1, \dots, \alpha_c$  in  $\mathbb{N}^r$  such that:*

- (1) *each  $\alpha_i$  has at most two nonzero entries;*
- (2) *the sum of the entries of each  $\alpha_i$  is  $p^e - 1$ ;*
- (3)  *$\beta_j = \sum_i \alpha_{ij}$ , where  $\alpha_i = (\alpha_{i1}, \dots, \alpha_{ic})$ .*

*Proof.* We will induce on  $r$ . If  $c = r = 1$ , then  $\beta = (p^e - 1)$  and we let  $\alpha_1 = \beta$ . If  $c = 1, r = 2$ , we have  $\beta = (\beta_1, \beta_2)$  where  $\beta_1 + \beta_2 = p^e - 1$  and we can let  $\alpha_1 = (\beta_1, \beta_2)$  and again (1)-(3) hold.

If  $c = r$  and  $\beta_1 = \dots = \beta_c = p^e - 1$ , then we can set  $\alpha_i$  to be the vector with  $p^e - 1$  at  $i$ -th spot and 0 elsewhere. Otherwise, there must be a  $\beta_i < p^e - 1$ . Without loss of generality, we assume that  $\beta_r < p^e - 1$ .

We claim that  $\beta_j \geq p^e - 1 - \beta_r$  for some  $j$  between 1 and  $r - 1$ , and we reason as follows. If  $c = r$ , then there must be a  $j$  such that  $\beta_j > p^e - 1$ , and hence  $\beta_j \geq p^e - 1 - \beta_r$ . Now assume that  $c = r - 1$ . Suppose  $\beta_i < p^e - 1 - \beta_r$  for all  $i \leq r - 1$ , as then we would have:

$$\sum_{i=1}^r \beta_i < (r - 1)(p^e - 1 - \beta_r) + \beta_r \leq (r - 2)(p^e - 1 - \beta_r) + (p^e - 1) \leq (r - 1)(p^e - 1) = c(p^e - 1)$$

which contradicts the assumption that  $\sum_{i=1}^r \beta_i = c(p^e - 1)$ . So, there is a  $j$  between 1 and  $r - 1$  such that  $\beta_j \geq p^e - 1 - \beta_r$ .

Set  $\alpha_c := (0, \dots, 0, p^e - 1 - \beta_r, 0, \dots, \beta_r)$  where  $p^e - 1 - \beta_r$  appears in the  $j$ -th spot. Consider

$$(\beta_1, \dots, \beta_{j-1}, \beta_j - (p^e - 1 - \beta_r), \beta_{j+1}, \dots, \beta_{r-1}).$$

This is an element of  $\mathbb{N}^{r-1}$  such that the sum of its entries is  $(c - 1)(p^e - 1)$ . By our induction hypotheses, there are  $\gamma_1, \dots, \gamma_{c-1} \in \mathbb{N}^{r-1}$  that satisfy (1), (2), and (3). For  $1 \leq i \leq c - 1$ , setting  $\alpha_i$  be  $\gamma_i$  with a 0 added to the end completes the proof of our lemma.  $\square$

*Proof of Theorem 3.3.* By Remark 3.2,  $J$  is a minimal prime of  $(g_1, \dots, g_c)$ . Hence once we identify

$$\omega_{S/J} = \text{Hom}_{S/(g_1, \dots, g_c)}(S/J, S/(g_1, \dots, g_c)) = ((g_1, \dots, g_c) : J) \cdot (S/J) = I \cdot (S/J),$$

we know from Lemma 2.3 that

$$\Phi_{S/J}^e(F_*^e(-)) = \Phi_{S/(g_1, \dots, g_c)}^e(F_*^e(-))|_{\omega_{S/J}} = \text{Tr}_S^e(F_*^e(g_1^{p^e-1} \dots g_c^{p^e-1} \cdot -))|_{I \cdot (S/J)}.$$

Next we notice that for every  $1 \leq k \leq c$ ,  $(S/J)_{f_k} \cong R_{f_k}[u_{ij} | j \neq k]$  is regular. Hence for  $N \gg 0$ ,  $f_k^N$  is a test element for  $S/J$ . Thus by Remark 2.10, we have:

$$(3.5.1) \quad \tau(\omega_{S/J}) = \sum_e \Phi_{S/J}^e(F_*^e(f_k^N \cdot \omega_{S/J})) = \sum_{e \geq 0} \text{Tr}_S^e(F_*^e(g_1^{p^e-1} \dots g_c^{p^e-1} \cdot f_k^N \cdot I S)) \cdot (S/J)$$

Since  $f_k \in I$  and  $R$  is regular, by Remark 2.9, for  $N \gg 0$  we also have:

$$(3.5.2) \quad \tau(I^c) \cdot (S/J) = \sum_{e \geq 0} \text{Tr}_R^e(F_*^e((f_1, \dots, f_r)^{c(p^e-1)} \cdot f_k^N \cdot R)) \cdot (S/J)$$

When we expand  $g_1^{p^e-1} \dots g_c^{p^e-1}$ , it is easy to see from (3.5.1) that  $\tau(\omega_{S/J})$  can be generated by elements of the form

$$(3.5.3) \quad \text{Tr}_S^e \left( F_*^e \left( \binom{p^e-1}{\alpha_{11}, \dots, \alpha_{1r}} \dots \binom{p^e-1}{\alpha_{c1}, \dots, \alpha_{cr}} f_1^{\beta_1} f_2^{\beta_2} \dots f_r^{\beta_r} \prod u_{ij}^{\alpha_{ij}} \cdot f_k^N \cdot s \cdot \prod u_{ij}^{\gamma_{ij}} \right) \right)$$

where  $0 \leq \alpha_{ij} \leq p^e - 1$ ,  $\beta_j = \sum_{i=1}^c \alpha_{ij}$ ,  $\sum_{j=1}^t \beta_j = c(p^e - 1)$  and  $s \in I$ . By definition of the trace map, this is equal to

$$\prod u_{ij}^{\frac{\alpha_{ij} + \gamma_{ij} - (p^e - 1)}{p^e}} \cdot \text{Tr}_R^e \left( F_*^e \left( \binom{p^e-1}{\alpha_{11}, \dots, \alpha_{1r}} \dots \binom{p^e-1}{\alpha_{c1}, \dots, \alpha_{cr}} f_1^{\beta_1} f_2^{\beta_2} \dots f_r^{\beta_r} \cdot f_k^N \cdot s \right) \right)$$

where  $\frac{\alpha_{ij} + \gamma_{ij} - (p^e - 1)}{p^e}$  denotes 0 if  $\alpha_{ij} + \gamma_{ij} \not\equiv -1 \pmod{p^e}$ . But it is clear that this element is in  $\tau(I^c) \cdot S$  by expression (3.5.2). This proves (1).

Next we prove (2). By Lemma 3.4 we can assume that  $\tilde{I} = (f_1, \dots, f_{c+1})$  is a reduction of  $I$  (the case that  $I$  has a reduction generated by  $c$  elements is similar). Hence by the arguments above, we have that, for  $1 \leq k \leq c$  and  $N \gg 0$ ,

$$\tau(I^c) \cdot (S/J) = \tau(\tilde{I}^c) \cdot (S/J) = \sum_{e \geq 0} \text{Tr}_R^e(F_*^e((f_1, \dots, f_{c+1})^{c(p^e-1)} \cdot f_k^{N+1} \cdot R)) \cdot (S/J)$$

Given a generator  $f_1^{\beta_1} \dots f_{c+1}^{\beta_{c+1}}$  of  $(f_1, \dots, f_{c+1})^{c(p^e-1)}$ , we can find  $\alpha_1, \dots, \alpha_c \in \mathbb{N}^{c+1}$  satisfying the conclusion of Lemma 3.5. Then

$$\prod_{i,j} (u_{ij} f_j)^{\alpha_{ij}} = \prod_{i,j} u_{ij}^{\alpha_{ij}} \prod f_j^{\beta_j}$$

appears with coefficient  $\binom{p^e-1}{\alpha_{1,1}, \dots, \alpha_{1,c+1}} \dots \binom{p^e-1}{\alpha_{c,1}, \dots, \alpha_{c,c+1}}$  in the product  $g_1^{p^e-1} \dots g_c^{p^e-1}$ . Because each multinomial coefficient  $\binom{p^e-1}{\alpha_{i,1}, \dots, \alpha_{i,c+1}} = \binom{p^e-1}{\alpha_{i,j_i}}$  for some  $j_i$  by Lemma 3.5 (1)-(2), they are nonzero in  $k$ .

Each  $\alpha_{i,j}$  is less than  $p^e$ , so let

$$s' = \left( \prod_{\substack{1 \leq i \leq c \\ 1 \leq j \leq c+1}} u_{i,j}^{p^e-1-\alpha_{i,j}} \right) \left( \prod_{\substack{1 \leq i \leq c \\ c+2 \leq j \leq r}} u_{i,j}^{p^e-1} \right).$$

Then  $\text{Tr}_S^e(F_*^e(- \cdot s'))$  sends  $\prod_{i,j} u_{i,j}^{\alpha_{i,j}}$  to 1 and all other basis elements of the form  $\prod_{i,j} u_{i,j}^{\gamma_{i,j}}$  to 0. Hence,

$$\begin{aligned} & \text{Tr}_S^e(F_*^e(g_1^{p^e-1} \dots g_c^{p^e-1} \cdot f_k^{N+1} \cdot s' \cdot R)) \\ &= \text{Tr}_S^e \left( F_*^e \left( \binom{p^e-1}{\alpha_{1,1}, \dots, \alpha_{1,c+1}} \dots \binom{p^e-1}{\alpha_{c,1}, \dots, \alpha_{c,c+1}} \cdot \prod_{\substack{1 \leq i \leq c \\ 1 \leq j \leq r}} u_{i,j}^{p^e-1} \prod_{j=1}^{c+1} f_j^{\beta_j} \cdot f_k^{N+1} R \right) \right) \\ &= \text{Tr}_R^e \left( F_*^e \left( \prod_{j=1}^{c+1} f_j^{\beta_j} \cdot f_k^{N+1} \cdot R \right) \right). \end{aligned}$$

In particular,

$$\text{Tr}_R^e \left( F_*^e \left( \prod_{j=1}^{c+1} f_j^{\beta_j} \cdot f_k^{N+1} \cdot R \right) \right) \cdot (S/J) = \text{Tr}_S^e(F_*^e(g_1^{p^e-1} \dots g_c^{p^e-1} \cdot f_k^N \cdot f_k s' R)) \cdot (S/J) \subseteq \tau(\omega_{S/J})$$

for every generator  $\prod_{j=1}^{c+1} f_j^{\beta_j}$  of  $(f_1, \dots, f_{c+1})^{c(p^e-1)}$ , where the second inclusion follows from expression (3.5.1). Therefore we have

$$\begin{aligned}\tau(I^c) \cdot (S/J) &= \tau(\tilde{I}^c) \cdot (S/J) \\ &= \sum_{e \geq 0} \text{Tr}_R^e(F_*^e((f_1, \dots, f_{c+1})^{c(p^e-1)} \cdot f_k^{N+1} \cdot R)) \cdot (S/J) \\ &\subseteq \tau(\omega_{S/J}).\end{aligned}$$

□

*Remark 3.6.* The proof of Theorem 3.3 (2) requires the minimal reduction be generated by at most  $c+1$  elements. If not, then we are not in the case of Lemma 3.5 and it may be the case that there are always at least three nonzero entries in some  $\alpha_i$ . Consequently, multinomial coefficients must be taken into consideration.

**Corollary 3.7.** *With the notation as in Definition 3.1 and the assumptions as in Theorem 3.3 (2),  $\tau(\omega_{S/J}) = \omega_{S/J}$  if and only if  $\tau(I^c) = I$ . In particular,  $S/J$  has  $F$ -rational singularities if and only if  $S/J$  is Cohen-Macaulay and  $\tau(I^c) = I$ .*

*Proof.* If  $\tau(I^c) = I$ , then Theorem 3.3 immediately implies  $\tau(\omega_{S/J}) = \omega_{S/J}$ .

Conversely, assume that  $\tau(I^c) \neq I$  and  $\tau(\omega_{S/J}) = \omega_{S/J}$ . Since  $\tau(I^c)$  is always contained in  $I$  by (2.12.1), at least one of the generators of  $I$  is not in  $\tau(I^c)$ , say  $f_1$ . From Theorem 3.3, we can see that  $\tau(I^c)S + J = IS + J$ ; hence  $f_1 \in \tau(I^c)S + J$ . Thus, there are elements  $a \in \tau(I^c)S$  and  $b \in J$  such that  $f_1 = a + b$ . (Note that  $b \neq 0$ .) Then we have  $f_1 - a \in J$  which implies that  $(f_1 - a)f_1 \in (g_1, \dots, g_c)$ . However, this is impossible because of the degrees in the  $u_{ij}$ . This is a contradiction.

The last assertion is clear because  $S/J$  is  $F$ -rational if and only if  $S/J$  is Cohen-Macaulay and  $\tau(\omega_{S/J}) = \omega_{S/J}$ . □

**Corollary 3.8.** *With the notation as in Definition 3.1 and the assumptions as in Theorem 3.3 (2), if the pair  $(R, I^c)$  is  $F$ -pure and  $R/I$  is Cohen-Macaulay, then  $S/J$  is  $F$ -rational. In particular, if  $R/I$  is an  $F$ -pure complete intersection, then  $S/J$  is  $F$ -rational.*

*Proof.* By Lemma 2.12,  $(R, I^c)$  is  $F$ -pure implies  $\tau(I^c) = I$ . The first statement thus follows from Corollary 3.7. Finally, it is well known that when  $R/I$  is an  $F$ -pure complete intersection, the pair  $(R, I^c)$  is  $F$ -pure. This follows from a Fedder type criterion ([Tak04, Lemma 3.9] and others). □

We can recover [Niu14, Corollary 3.4] in the complete intersection and almost complete intersection cases.

**Corollary 3.9.** *Let  $I = (f_1, \dots, f_r)$  be an ideal of  $\mathbb{C}[x_1, \dots, x_n]$  and let  $c$  be the codimension of  $I$ . Let  $S$  and  $J$  be in Definition 3.1. Assume that  $r \leq c+1$ . Then  $S/J$  has rational singularities if and only if  $S/J$  is Cohen-Macaulay and  $\mathcal{J}(I^c) = I$ , where  $\mathcal{J}(I^c)$  is the multiplier ideal of  $I^c$ .*

*Proof.* By [Smi97] and [Har98],  $S/J$  has rational singularities if and only if its reduction  $(S/J)_p$  is  $F$ -rational for all  $p \gg 0$ . It is easy to see that, for  $p \gg 0$ , the reduction  $J_p$  of  $J$  is a generic link of the reduction  $I_p$  of  $I$ . Hence,  $S/J$  has rational singularities if and only if  $(S/J)_p$  is Cohen-Macaulay and  $\tau(I_p^c) = I_p$  for  $p \gg 0$  by Corollary 3.7. On the other hand, it was proved in [HY03] that  $\mathcal{J}(I^c)_p = \tau(I_p^c)$  for all  $p \gg 0$ . Therefore, we have  $S/J$  has rational singularities if and only if  $(S/J)_p$  is Cohen-Macaulay and  $\mathcal{J}(I^c)_p = I_p$  for  $p \gg 0$ . This completes the proof. □

#### 4. BEHAVIOR OF $F$ -PURE THRESHOLD UNDER GENERIC LINKAGE

In this section we investigate behaviors of  $F$ -pure thresholds under generic linkages. We begin with an easy lemma.

**Lemma 4.1.** *Let  $R = k[x_1, \dots, x_n]$  be a polynomial ring over a perfect field of characteristic  $p$  and  $I$  be an unmixed ideal of  $R$ . Let  $\Lambda_1$  and  $\Lambda_2$  be 2 sets of generators of  $I$  and let  $(S_i, J_i)$  be the generic link with respect to  $\Lambda_i$  ( $i=1,2$ ). Then*

$$\text{fpt}_{S_1}(J_1) = \text{fpt}_{S_2}(J_2).$$



*Proof.* As in the proof of Lemma 3.4, we can assume that  $\Lambda_1 = \{f_1, \dots, f_r\}$  and  $\Lambda_2 = \{f_1, \dots, f_r, f_{r+1}\}$ . Let  $\varphi$  and  $S_2^\varphi$  be the same as in the proof of Lemma 3.4. It is straightforward to check that

$$\tau(J_1^t) \otimes_{S_1} S_2^\varphi = \tau(J_2^t)$$

for each nonnegative real number  $t$ . Our lemma follows immediately.  $\square$

**Remark 4.2.** Let  $k \subseteq K$  be an extension of perfect fields and let  $R = k[x_1, \dots, x_n]$  and  $T = K[x_1, \dots, x_n]$ . Since  $\text{Hom}_R(R^{1/p^e}, R)$  and  $\text{Hom}_T(T^{1/p^e}, T)$  are generated by the same projection, we have  $\tau_R(I^t) = \tau_T((IT)^t)$  (c.f. [BMS08, Remark 2.18]).

**Theorem 4.3.** *Let  $R = k[x_1, \dots, x_n]$  be a polynomial ring over a perfect field of characteristic  $p$  and  $I$  be an unmixed ideal of height  $c$  in  $R$ . Assume that  $I = (f_1, \dots, f_s)$  and that  $I$  has a reduction  $\tilde{I}$  generated by  $r$  elements. Let  $S = R[u_{i,j}]_{1 \leq i \leq c, 1 \leq j \leq s}$  be a polynomial ring over  $R$ . For  $1 \leq i \leq c$ , let*

$$g_i = u_{i,1}f_1 + u_{i,2}f_2 + \dots + u_{i,s}f_s.$$

*Then  $\text{fpt}_S(g_1, \dots, g_c) \geq \frac{c}{r} \text{fpt}_R(I)$ .*

*Proof.* By Lemma 4.1, we can add the generators of  $\tilde{I}$  to those of  $I$  and then assume that  $\tilde{I} = (f_1, \dots, f_r)$ . Since  $\tilde{I}$  is a reduction of  $I$ , it follows from [TW04, Proposition 2.2(6)] that  $\text{fpt}_R(I) = \text{fpt}_R(\tilde{I})$ . Hence it suffices to show that  $\tau_R(\tilde{I}^t) = R$  implies  $\tau_S((g_1, \dots, g_c)^{\frac{ct}{r}}) = S$  for a positive real number  $t$ . To this end, assume that  $\tau_R(\tilde{I}^t) = R$ . By Remark 4.2, we may assume that  $k$  is algebraically closed.

We wish to show that  $\tau_S((g_1, \dots, g_c)^{\frac{ct}{r}}) = S$ . Suppose otherwise and we seek a contradiction. There is a maximal ideal  $\mathfrak{m}$  of  $S$  such that  $\tau_S((g_1, \dots, g_c)^{\frac{ct}{r}}) \subseteq \mathfrak{m}$ . Since  $k$  is algebraically closed, we can write  $\mathfrak{m} = (x_1 - a_1, \dots, x_n - a_n, u_{11} - b_{11}, \dots, u_{cr} - b_{cr})$  for some  $a_i, b_{ij} \in k$ . Set  $\mathfrak{n} = (x_1 - a_1, \dots, x_n - a_n)$ . Since  $\tau_R(\tilde{I}^t) = R$ , there exist an integer  $e$ , an  $R$ -linear map  $\phi \in \text{Hom}_R(R^{1/p^e}, R)$ , and nonnegative integers  $a_1, \dots, a_r$  with  $\sum_i a_i = \lceil tp^e \rceil$  such that  $\phi(f_1^{a_1/p^e} \dots f_r^{a_r/p^e}) \notin \mathfrak{n}$ .

Without loss of generality, we may assume that  $a_1 \geq a_2 \geq \dots \geq a_r$ . Consequently,

$$a_1 + \dots + a_c \geq \left\lceil \frac{c}{r}(a_1 + \dots + a_r) \right\rceil = \left\lceil \frac{c}{r} \lceil tp^e \rceil \right\rceil \geq \left\lceil \frac{c}{r} tp^e \right\rceil$$

Let  $\phi_{\underline{a}} = \phi(f_{c+1}^{a_{c+1}/p^e} \dots f_r^{a_r/p^e} \cdot -)$ , i.e. pre-multiplication by  $f_{c+1}^{a_{c+1}/p^e} \dots f_r^{a_r/p^e}$  followed by the application of  $\phi$ . It is clear that  $\phi_{\underline{a}} : R^{1/p^e} \rightarrow R$  is an  $R$ -linear map and that  $\phi_{\underline{a}}(f_1^{a_1/p^e} \dots f_c^{a_c/p^e}) \notin \mathfrak{n}$ . We can extend  $\phi_{\underline{a}}$  to an  $S$ -linear map  $\psi_{\underline{a}} : R^{1/p^e}[u_{ij}] \rightarrow S = R[u_{ij}]$  that sends each  $u_{ij}$  to itself and restricts to  $\phi_{\underline{a}}$  on  $R^{1/p^e}$ .

It is clear that  $S^{1/p^e} = R^{1/p^e}[u_{ij}^{1/p^e}]$  is a free  $R^{1/p^e}[u_{ij}]$ -module with a basis  $\{\prod_{0 \leq b_{ij} \leq p^e-1} u_{ij}^{b_{ij}/p^e}\}$ . Let  $\pi_{\underline{a}} : R^{1/p^e}[u_{ij}^{1/p^e}] \rightarrow R^{1/p^e}[u_{ij}]$  be the projection that sends  $u_{11}^{a_1/p^e} \dots u_{cc}^{a_c/p^e}$  to 1 and all other basis element to 0.

Let  $\theta_{\underline{a}}$  be the composition of  $S^{1/p^e} \xrightarrow{\pi_{\underline{a}}} R^{1/p^e}[u_{ij}] \xrightarrow{\psi_{\underline{a}}} S$ . It is clear that  $\theta_{\underline{a}}$  is  $S$ -linear. By the construction of  $\pi_{\underline{a}}$ , it is straightforward to check that

$$\theta_{\underline{a}}(g_1^{a_1/p^e} \dots g_c^{a_c/p^e}) = \theta_{\underline{a}}((u_{11}f_1)^{a_1/p^e} \dots (u_{cc}f_c)^{a_c/p^e}) = \phi(f_1^{a_1/p^e} \dots f_r^{a_r/p^e}).$$

Since  $\phi(f_1^{a_1/p^e} \dots f_r^{a_r/p^e})$  is in  $R$  but not in  $\mathfrak{n} = (x_1 - a_1, \dots, x_n - a_n)$ , we must have

$$\phi(f_1^{a_1/p^e} \dots f_r^{a_r/p^e}) \notin \mathfrak{m} = (x_1 - a_1, \dots, x_n - a_n, u_{11} - b_{11}, \dots, u_{cr} - b_{cr}),$$

a contradiction to the assumption that  $\tau_S((g_1, \dots, g_c)^{\frac{ct}{r}}) \subseteq \mathfrak{m}$  (note that  $g_1^{a_1} \dots g_c^{a_c} \in (g_1, \dots, g_c)^{\lceil \frac{ct}{r} p^e \rceil}$ ).  $\square$

We have some immediate corollaries.

**Corollary 4.4.** *Let  $R = k[x_1, \dots, x_n]$  be a polynomial ring over a perfect field of characteristic  $p$  and  $I$  be an unmixed ideal of height  $c$  in  $R$ . Let  $J$  be a generic link of  $I$  in  $S = R[u_{ij}]$ . The following hold:*

- (1) *If  $I$  has a reduction generated by  $r$  elements, then  $\text{fpt}_S(J) \geq \frac{c}{r} \text{fpt}_R(I)$ .*
- (2) *If  $I$  has a reduction generated by  $c$  elements, in particular if  $I$  is a complete intersection, then  $\text{fpt}_S(J) \geq \text{fpt}_R(I)$ .*
- (3)  *$\text{fpt}_S(J) \geq \frac{c}{n} \text{fpt}_R(I)$  (note  $n = \dim(R)$ ).*

*Proof.* To prove (1), note that since  $(g_1, \dots, g_c) \subseteq J$ , we have  $\text{fpt}_S(J) \geq \text{fpt}_S(g_1, \dots, g_c)$ . Theorem 4.3 then completes the proof.

(2) is an immediate consequence of (1).

(3) By Remark 4.2, passing to the algebraic closure of  $k$  doesn't affect  $\text{fpt}_R(I)$  and  $\text{fpt}_S(J)$ . Hence we can assume that  $k$  is algebraically closed and hence is infinite. [Lyu86, Theorem] asserts that each ideal  $I$  admits a reduction generated by  $n$  elements. We are done by (1).  $\square$

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